

# ON THE SINGULAR BRAID MONOID

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ABSTRACT. Garside's results and the existence of the greedy normal form for braids are shown to be true for the singular braid monoid. An analogue of the presentation of J. S. Birman, K. H. Ko and S. J. Lee for the braid group is also obtained for this monoid.

## 1. INTRODUCTION

Various questions concerning braid groups and their generalizations attracted attention during the last decade. Presentations of the braid groups and algorithmic problems for the singular braid monoid are among these questions

The canonical presentation of the braid group  $Br_n$  was given by E. Artin [2] and is well known. It has the generators  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ , and relations

$$(1) \quad \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & \text{if } |i - j| > 1, \quad i, j = 1, \dots, n-1; \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, \dots, n-2. \end{cases}$$

Of course, there exist other presentations of the braid group. J. S. Birman, K. H. Ko and S. J. Lee [5] introduced a presentation with generators  $a_{ts}$  with  $1 \leq s < t \leq n$ , and relations

$$(2) \quad \begin{cases} a_{ts} a_{rq} = a_{rq} a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts} a_{sr} = a_{tr} a_{ts} = a_{sr} a_{tr} & \text{for } 1 \leq r < s < t \leq n. \end{cases}$$

The generators  $a_{ts}$  are expressed by canonical generators  $\sigma_i$  in the following form:

$$a_{ts} = (\sigma_{t-1} \sigma_{t-2} \cdots \sigma_{s+1}) \sigma_s (\sigma_{s+1}^{-1} \cdots \sigma_{t-2}^{-1} \sigma_{t-1}^{-1}) \quad \text{for } 1 \leq s < t \leq n.$$

The *Baez–Birman monoid*  $SB_n$  or *singular braid monoid* [3], [4] is defined as a monoid with generators  $\sigma_i, \sigma_i^{-1}, x_i, i = 1, \dots, n-1$ , and relations

$$(3) \quad \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & \text{if } |i - j| > 1, \\ x_i x_j = x_j x_i, & \text{if } |i - j| > 1, \\ x_i \sigma_j = \sigma_j x_i, & \text{if } |i - j| \neq 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_{i+1} x_i = x_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_{i+1} \sigma_i x_{i+1} = x_i \sigma_{i+1} \sigma_i, \\ \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1. \end{cases}$$

In pictures  $\sigma_i$  corresponds to the canonical generator of the braid group and  $x_i$  represents an intersection of the  $i$ th and  $(i+1)$ st strand as in Figure 1. Motivation for introduction of this object was Vassiliev – Goussarov theory of finite type invariants.

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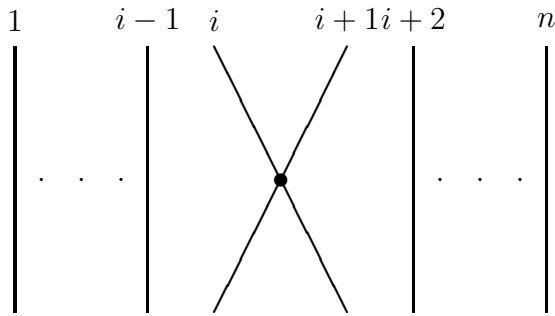


FIGURE 1.

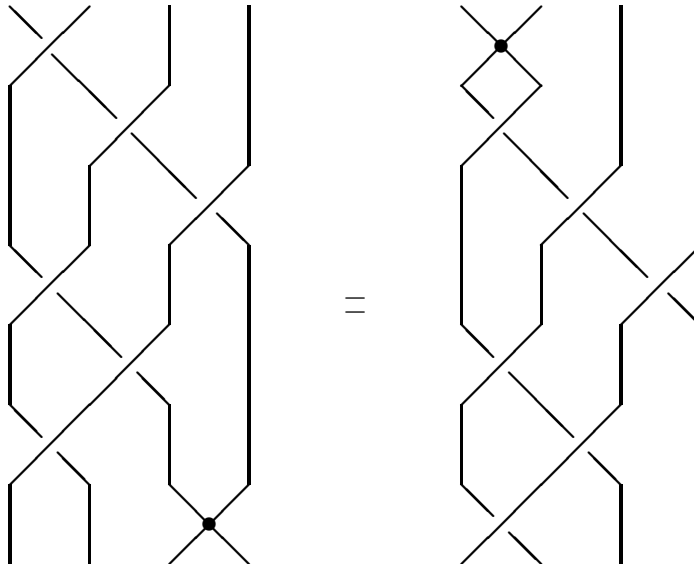


FIGURE 2.

The singular braid monoid on two strings is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}^+$ , so the word problem in this case is trivial. For the monoid with three strings this problem was solved by A. J  rai [12] and O. T. Dashbach and B. Gemein [7]. In general it was done by R. Corran in the complicated and technical paper [6]. Here we hope to give a simple solution using almost nothing but Garside's arguments. The initial idea is simple and geometric: Garside's *fundamental word*  $\Delta$  (for convenience the definition is given below) behaves the same way with respect to the singular generators  $x_i$  as it behaves with respect to the braid generators  $\sigma_i$ :

$$x_i \Delta \doteq \Delta x_{n-i},$$

as shown on Figure 2.

## 2. WORD PROBLEM FOR THE SINGULAR BRAID MONOID

Following Garside's ideas we consider the *positive singular braid monoid*  $SB_n^+$ . It is defined as a monoid with generators  $\sigma_i, x_i, i = 1, \dots, n-1$ , and relations (3) except the last one concerning the invertibility of  $\sigma_i$ . Two positive words  $A$  and  $B$  in the alphabet  $\{\sigma_i, x_i, (i = 1, \dots, n-1)\}$  will be said to be *positively equal* if they are equal as elements of  $SB_n^+$ . In this case we shall write  $A \doteq B$ . As usual, identity of words is denoted by the symbol  $\equiv$ . Proofs of statements

below are the same as in Garside's paper [11] with some exceptions as, for example, the proof of Proposition 5. Garside's proof of this Proposition doesn't work and we use the Malcev rule. Proposition 5 was proved by R. Corran [6], Theorem 3 was proved by R. Fenn, D. Rolfsen and J. Zhu [10] using different methods.

**Proposition 1.** *For  $i, k = 1, \dots, n-1$ , given  $\sigma_i A \doteq \sigma_k B$ , it follows that*

*if  $k = i$ , then  $A \doteq B$ ,*

*if  $|k - i| = 1$ , then  $A \doteq \sigma_k \sigma_i Z$ ,  $B \doteq \sigma_i \sigma_k Z$  for some  $Z$ ,*

*if  $|k - i| \geq 2$ , then  $A \doteq \sigma_k Z$ ,  $B \doteq \sigma_i Z$  for some  $Z$ ,*

*given  $\sigma_i A \doteq x_k B$ , it follows that*

*if  $|k - i| = 1$ , then  $A \doteq \sigma_k x_i Z$ ,  $B \doteq \sigma_i \sigma_k Z$  for some  $Z$ ,*

*if  $|k - i| \neq 1$ , then  $A \doteq x_k Z$ ,  $B \doteq \sigma_i Z$  for some  $Z$ .*

*and given  $x_i A \doteq x_k B$ , it follows that*

*if  $k = i$ , then  $A \doteq B$ ,*

*if  $|k - i| \geq 2$ , then  $A \doteq x_k Z$ ,  $B \doteq x_i Z$  for some  $Z$ ,*

*the case when  $|k - i| = 1$ , is impossible.*

*The same is true for multiples of  $\sigma_i$  or  $x_k$  to the right.*

*Proof.* Garside's proof works here. We apply the induction on the length  $s$  of  $A$  and the length of chain of transformations from  $a_i A$  to  $a_k B$  where  $a_i$  may be  $\sigma_i$  or  $x_i$  and  $a_k$  may be  $\sigma_k$  or  $x_k$ . The cases of  $s = 0, 1$  are evident, so suppose that the statement is true for length  $s \leq r$  and for  $s = r + 1$  it is true for chain-length  $\leq t$ . As an example we give a proof of the last statement, which is formally is not contained in Garside's considerations. So, let  $A, B$  be of word-length  $r + 1$  and let  $x_i A \doteq x_k B$ ,  $|i - k| = 1$ , through a transformation of chain-length  $t + 1$ . We may suppose that  $k = i + 1$  and let the successive words of transformations be

$$W_1 \equiv x_i A, \dots, W_{t+2} \equiv x_{i+1} B.$$

Choose arbitrary any intermediate word  $W_g$ , say, from the middle of the chain somewhere. We have  $W \equiv aV$ , where  $a$  is a generator of  $SB_n^+$ . Suppose at first that  $a$  commutes with  $x_i, x_{i+1}$ , then we have  $x_i A \doteq aV \doteq x_{i+1} B$  and using induction we obtain:

$$A \doteq aP, \quad V \doteq x_i P, \quad V \doteq x_{i+1} Q, \quad B \doteq aQ.$$

So,  $x_i P \doteq x_{i+1} Q$  what is impossible by induction.

The cases when  $a = x_{i-1}, x_i, x_{i+1}, x_{i+2}$  are also impossible by induction. The cases which we need to consider are  $a = \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}$ . So, let  $a = \sigma_{i-1}$ , then  $W_g \doteq \sigma_{i-1} V$ . Using induction we get

$$A \doteq \sigma_{i-1} \sigma_i P, \quad V \doteq \sigma_i x_{i-1} P, \quad V \doteq x_{i+1} Q, \quad B \doteq \sigma_{i-1} Q.$$

Hence  $\sigma_i x_{i-1} P \doteq x_{i+1} Q$ , and so,  $x_{i-1} P \doteq \sigma_{i+1} x_i R$ ,  $Q \doteq \sigma_i \sigma_{i+1} R$ . Again using induction we get  $P \doteq \sigma_{i+1} S$ ,  $x_i R \doteq x_{i-1} S$ , that is impossible. The rest cases may be considered the same way.  $\square$

**Corollary 1.** *If  $A \doteq P$ ,  $B \doteq Q$ ,  $AXB \doteq PYQ$ , ( $L(A) \geq 0$ ,  $L(B) \geq 0$ ), then  $X \doteq Y$ . That is, monoid  $SB_n$  is left and right cancellative.*

**Proposition 2.** *For any word  $W$  in the alphabet  $\{\sigma_i, x_i \ (i = 1, \dots, n-1)\}$ , let  $S$  be a word in the alphabet  $\{\sigma_i\}$  of maximal length, such that  $W \doteq ST$  for some word  $T$ . Suppose also that  $W \doteq AV$  for some word  $A$  in the alphabet  $\{\sigma_i\}$ . Then  $S$  is divisible by  $A$ . The same is true for the right division.*

*Proof.* We use the induction on the length of  $S$ . If  $S$  has the length 1, then  $A$  also has the length 1 and the assertion follows from Proposition 1. Let the statement be true for the length less or equal to  $k$  and let the length of  $S$  be equal to  $k + 1$ . Consider at first the case when the length of  $A$  is equal to 1. This means that  $A \equiv \sigma_j$ . Let the first letter of  $S$  be  $\sigma_i$ :  $S \equiv \sigma_i S'$ , so we have the situation  $\sigma_i S' T \doteq \sigma_j V$ . If  $i = j$ , then we are done. If  $|i - j| \geq 2$ , then  $S' T \doteq \sigma_j X$  and using the induction we have  $S' \doteq \sigma_j R'$ . Let  $|i - j| = 1$ , then  $S' T \doteq \sigma_j \sigma_i Y$  and again using induction we obtain  $S' \doteq \sigma_j \sigma_i Q'$ . Suppose now that the length of  $A$  is greater than 1, then  $A \equiv \sigma_j A'$  for some  $\sigma_j$ . Using the previous case we have  $S \doteq \sigma_j S'$  and  $S' T \doteq A' V$  and by induction we obtain  $S' \doteq A' S''$ . Hence  $S$  is divisible by  $A$ .  $\square$

**Corollary 2.** *For any element  $w$  of  $SB_n^+$  there exists a unique greatest left (right) divider which belongs to  $Br_n^+$ .*

Garside's *fundamental word* for the braid group  $Br_n$  is the following

$$\Delta \equiv \sigma_1 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1.$$

If we use Garside's notation  $\Pi_t \equiv \sigma_1 \dots \sigma_t$ , then  $\Delta \equiv \Pi_{n-1} \dots \Pi_1$ . We keep the same notations for their images in  $SB_n$ . Garside's transformation of words  $\mathcal{R}$  and then the automorphism of  $Br_n$  and the positive braid monoid  $Br_n^+$ , defined by the formula

$$\mathcal{R}(\sigma_i) \equiv \sigma_{n-i},$$

we extend to letters  $x_i$  and so to  $SB_n^+$  and to  $SB_n$  by

$$\mathcal{R}(x_i) \equiv x_{n-i}.$$

**Proposition 3.** *There are equalities*

$$\sigma_i \Delta \doteq \Delta \mathcal{R}(\sigma_i),$$

$$x_i \Delta \doteq \Delta \mathcal{R}(x_i).$$

*Proof.*

$$\begin{aligned} x_1 \Delta &\equiv x_1 \sigma_1 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \sigma_1 x_1 \sigma_2 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \\ &\sigma_1 x_1 \sigma_2 \sigma_1 \dots \sigma_{n-1} \sigma_2 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \sigma_1 \sigma_2 \sigma_1 x_2 \sigma_3 \dots \sigma_{n-1} \sigma_2 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \\ &\sigma_1 \sigma_2 \sigma_1 \sigma_3 \dots x_{n-2} \sigma_{n-1} \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \sigma_1 \sigma_2 \sigma_1 \sigma_3 \dots \sigma_{n-2} \sigma_{n-1} x_{n-1} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \\ &\sigma_1 \sigma_2 \sigma_3 \dots \sigma_{n-2} \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 x_{n-1} \equiv \Delta x_{n-1}. \end{aligned}$$

Let  $i \geq 2$ :

$$\begin{aligned} x_i \Delta &\equiv x_i \sigma_1 \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \sigma_1 \dots x_i \sigma_{i-1} \sigma_i \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \\ &\sigma_1 \dots \sigma_{i-1} \sigma_i x_{i-1} \dots \sigma_{n-1} \sigma_1 \dots \sigma_{n-2} \dots \sigma_1 \sigma_2 \sigma_1 \doteq \Pi_{n-1} x_{i-1} \Pi_{n-2} \dots \Pi_1 \doteq \\ &\Pi_{n-1} \dots \Pi_{n-i+1} x_1 \Pi_{n-i} \Pi_{n-i-1} \dots \Pi_1 \doteq \Pi_{n-1} \dots \Pi_{n-i+1} \Pi_{n-i} \Pi_{n-i-1} x_{n-i} \Pi_{n-i-2} \dots \Pi_1 \doteq \\ &\Pi_{n-1} \dots \Pi_{n-i+1} \Pi_{n-i} \dots \Pi_1 x_{n-i} \doteq \Delta x_{n-i}. \end{aligned}$$

$\square$

**Proposition 4.** *If  $W$  is any positive word in  $SB_n^+$  such that either*

$$W \doteq \sigma_1 A_1 \doteq \sigma_2 A_2 \doteq \dots \doteq \sigma_{n-1} A_{n-1},$$

*or*

$$W \doteq B_1 \sigma_1 \doteq B_2 \sigma_2 \doteq \dots \doteq B_{n-1} \sigma_{n-1},$$

*then  $W \doteq \Delta Z$  for some  $Z$ .*

**Proposition 5.** *Canonical homomorphism*

$$SB_n^+ \rightarrow SB_n$$

*is a monomorphism.*

*Proof.* We need to prove that if two elements of  $SB_n^+$ , (expressed by positive words  $A$  and  $C$ ) are equal in  $SB_n$  then they are positively equal. If elements defined by words  $A$  and  $B$  are equal in  $SB_n$  then there exists a sequence of words

$$(4) \quad A \equiv A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_j \rightarrow \dots \rightarrow A_k \equiv C,$$

where each arrow means an elementary operation which may be an application of one defining relation or insert or deletion of an expression  $cc^{-1}$  or  $c^{-1}c$  where  $c$  is one of  $\sigma_i$ . In the first case the insert is called *left* and in the second it is called *right*. A right insert

$$A_j \equiv Y_j Z_j \rightarrow Y_j c^{-1} c Z_j$$

is called *correct*, if the fragment  $Y_j$  is not changed in the sequence (4) till the elimination of  $c^{-1}$ . For the left insert the same condition is made for the fragment  $Z_j$ .

We use the following Malcev rule [13], [14], [15]:

*If all defining relations contain only positive degrees of letters of the alphabet then transformation of a word  $A$  into a positive word  $C$  can be done by a sequence of operations where all inserts are correct.*

Let us consider a subsequence of (4) starting with the last right insert and finishing with the deletion of the corresponding  $c^{-1}$ :

$$A_j \equiv Y_j Z_j \rightarrow Y_j c^{-1} c Z_j \rightarrow Y_j c^{-1} V_{j+1} \rightarrow \dots \rightarrow Y_j c^{-1} c Z_{j+r} \rightarrow Y_j Z_{j+r}.$$

The fragment  $Y_j$  does not change in this subsequence because the insert is correct. For any  $c$  there exists a positive word  $D_c$  such that  $D_c c \doteq \Delta$ . Define the sequence

$$\begin{aligned} \Delta A_j \equiv \Delta Y_j Z_j \mapsto \dots \mapsto \mathcal{R}(Y_j) \Delta Z_j \mapsto \dots \mapsto \mathcal{R}(Y_j) D_c c Z_j \hookrightarrow \mathcal{R}(Y_j) D_c V_{j+1} \hookrightarrow \dots \\ \hookrightarrow \mathcal{R}(Y_j) D_c c Z_{j+r} \mapsto \dots \mapsto \mathcal{R}(Y_j) \Delta Z_{j+r} \mapsto \dots \mapsto \Delta Y_j Z_{j+r} \end{aligned}$$

where  $\mapsto$  denote a positive operation and  $\hookrightarrow$  denote a positive operation or deletion. So multiplying by  $\Delta$  we eliminated one insert. Applying induction we construct a sequence of positive operations between  $\Delta^m A \Delta^l$  and  $\Delta^m C \Delta^l$  for some  $m$  and  $l$ . After the cancellation this means that  $A$  and  $C$  are positive equivalent. As it was noted by V. V. Chainikov, one can get a proof without use of the Malcev rule but using the fact that relations (3) are invariant with respect to the operation  $\mathcal{R}$ .  $\square$

Among all positive words on the alphabet  $\{\sigma_1, \dots, \sigma_{n-1}, x_1, \dots, x_{n-1}\}$  let us introduce a lexicographical ordering with the condition that  $\sigma_1 < \sigma_2 < \dots < \sigma_{n-1} < x_1 < x_2 < \dots < x_{n-1}$ . For a positive word  $W$  the *base* of  $W$  is the smallest positive word with respect to this ordering, which is positively equal to  $W$ . The base is uniquely determined. If a positive word  $A$  is prime to  $\Delta$ , then for the base of  $A$  the notation  $\overline{A}$  will be used.

**Theorem 1.** *In  $SB_n$  every word  $W$  can be expressed uniquely in the form  $\Delta^m \overline{A}$ , where  $m$  is an integer.*

*Proof.* First suppose  $P$  is any positive word. Among all positive words positively equivalent to  $P$  choose a word in the form  $\Delta^t A$  with  $t$  maximal. Then  $A$  is prime to  $\Delta$  and we have

$$P \doteq \Delta^t \overline{A}$$

Now let  $W$  be any word in  $SB_n$ . Then we may put

$$W \equiv W_1(c_1)^{-1} W_2(c_2)^{-1} \dots (c_k)^{-1} W_{k+1},$$

where each  $W_j$  is a positive word of length  $\geq 0$ , and  $c_l$  are generators  $\sigma_i$ , the only possible invertible generators. As it was already mentioned for each  $c_l$  there exists a positive word  $D_l$  such that  $c_l D_l \doteq \Delta$ , so that  $(c_l)^{-1} = D_l \Delta^{-1}$ , and hence

$$W = W_1 D_1 \Delta^{-1} W_2 D_2 \Delta^{-1} \dots W_k D_k \Delta^{-1} W_{k+1}.$$

Hence, moving the factors  $\Delta^{-1}$  to the left, we obtain  $W = \Delta^k P$ , where  $P$  is positive, so we can express it in the form  $\Delta^t \overline{A}$  and finally we get

$$(5) \quad W = \Delta^m \overline{A}.$$

It remains to show that the form (5) is unique. Suppose

$$(6) \quad \Delta^m \overline{A} = \Delta^p \overline{C}.$$

Let  $p < m$ , and  $m - p = t > 0$ . Then (6) gives  $\Delta^t \overline{A} = \overline{C}$ , what is impossible. So  $p = m$  and hence  $\overline{A} = \overline{B}$ . So from Proposition 5 we obtain  $\overline{A} \doteq \overline{C}$ , but the base is unique, hence  $\overline{A} \equiv \overline{C}$  and the uniqueness of the form (5) is established.  $\square$

The form of a word  $W$  established in this theorem we call the *Garside left normal form* and the index  $m$  we call the *power* of  $W$ . The same way the *Garside right normal form* is defined and the corresponding variant of Theorem 1 is true. The Garside normal form also gives a solution to the word problem in the braid group.

**Corollary 3.** *The necessary and sufficient condition that two words in  $SB_n$  are equal is that their Garside normal forms (left or right) are identical.*

Garside normal form for the braid groups was precised in the subsequent papers [1], [9], [8]. Namely, it was written in the *left-greedy form* (in the terminology of W. Thurston [9])

$$\Delta^t A_1 \dots A_k,$$

where  $A_i$  are the successive possible longest *fragments of the word*  $\Delta$  (in the terminology of S. I. Adyan [1]) or *positive permutation braids* (in the terminology of E. El-Rifai and H. R. Morton [8]). Certainly, the same way the *right-greedy form* is defined.

Consider these forms for the singular braid monoid. For any word  $W$  first of all move to the left the greatest power of  $\Delta$ . To the right we have a positive word  $W'$  not divisible by  $\Delta$ . Consider the decomposition  $W' \doteq S_1 T$  of Proposition 2. Then we take the fragments of the left-greedy form for  $S_1$ :  $S_{1,1} \dots S_{1,t}$ . Among all the  $x_i$ -divisors of  $T$  we choose the smallest in the lexicographical order:  $T \doteq x_{i_1} T_1$ . Consider the decomposition of Proposition 2 for  $T_1$  and continue this process. We get the form

$$W \doteq \Delta^t S_1 X_1 \dots S_k X_k,$$

where each  $S_i$  consists of the fragments of the left-greedy form for the braid group and each  $X_i$  is a lexicographically ordered product of  $x_j$ . We call this form the *left-greedy form* for the singular braid monoid. The same way the *right-greedy form* for the singular braid monoid is defined.

**Theorem 2.** *Each element of the singular braid monoid can be written uniquely in a left-greedy (right-greedy) form.*

**Theorem 3.** *For  $n = 2$  the singular braid monoid  $SB_n$  is commutative and isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}^+$ . For  $n \geq 3$  its center is the same as the center of  $Br_n$  and so is generated by  $\Delta^2$ .*

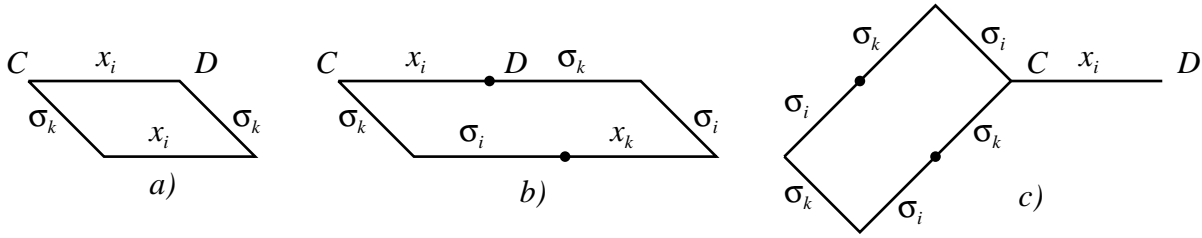


FIGURE 3.

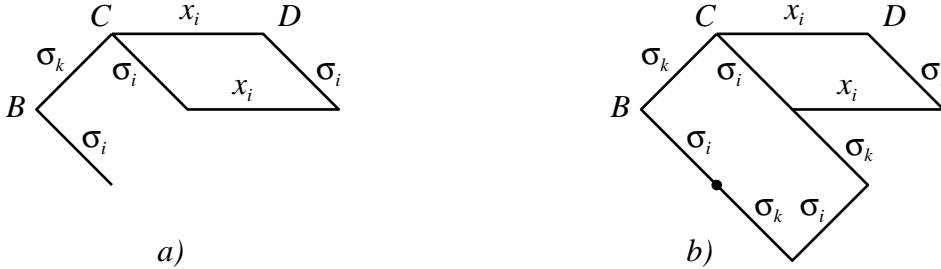


FIGURE 4.

### 3. CONJUGACY PROBLEM FOR THE SINGULAR BRAID MONOID

Let  $M$  be a monoid with unit group  $G$ . We call two elements  $u, v \in M$  *conjugate* if  $v = g^{-1}ug$  for some  $g \in G$ . This means that the elements  $u$  and  $v$  belong to the same orbit of the canonical action of the braid group  $Br_n$  on  $SB_n$ . So, in this sense we understand the *conjugacy problem* for the singular braid monoid. Such approach to conjugacy appears in monoid theory, see [16], for example.

Relations (3) are homogeneous with respect to both types of generators  $\sigma_i$  and  $x_i$ , so, three homomorphisms which express the degree of an element with respect to  $\sigma_i$ ,  $x_i$  and the total degree are well defined:  $\deg_\sigma : SB_n \rightarrow \mathbb{Z}$ ,  $\deg_x : SB_n \rightarrow \mathbb{Z}^+$  and  $\deg : SB_n \rightarrow \mathbb{Z}$ .

**Proposition 6.** *The group of units of monoid  $SB_n$  is equal to the image of the braid group  $Br_n$ .*

*Proof.* Invertible elements must have invertible degree  $\deg_x$ , so  $\deg_x = 0$ . □

Garside's solution of the conjugacy problem for the braid groups works in the case of the singular braid monoid. In the proof of the following theorem concerning the structure of the Cayley diagram  $D(W)$  of a positive word  $W$  on the alphabet  $\{\sigma_i, x_k\}$ , containing  $\Delta$ , some additional considerations are necessary.

**Theorem 4.** *If  $W \doteq \Delta V$  is any positive word (on the alphabet  $\sigma_i, x_i$ ) containing  $\Delta$ , then each node of  $D(W)$  is incident with each edge  $\sigma_i$ ,  $i = 1, \dots, n-1$ .*

*Proof.* To complete Garside's proof by induction on the order of a node we must consider the case when the edge  $x_i$  starts at some node  $C$  of order  $m$  and ends at a node  $D$  of order  $m+1$ .

We first consider the case of the generators  $\sigma_k$ , with  $|k-i| \neq 1$ . If the edge  $\sigma_k$  ends at  $C$  then we have  $\sigma_k x_i \doteq x_i \sigma_k$  what means that the edge  $\sigma_k$  also ends at  $D$ . If  $\sigma_k$  starts from  $C$  then using Proposition 1 we obtain the fragment of the Cayley graph, depicted on Figure 3 a).

Let  $|k-i| = 1$ . If the edge  $\sigma_k$  starts at  $C$ , then using Proposition 1 we obtain the fragment of the Cayley graph, depicted on Figure 3 b). Consider now the case when the edge  $\sigma_k$  ends at  $C$  (and starts at some node  $B$ ). The edge  $\sigma_i$  must be incident to  $C$ , so it either ends or starts at  $C$ .

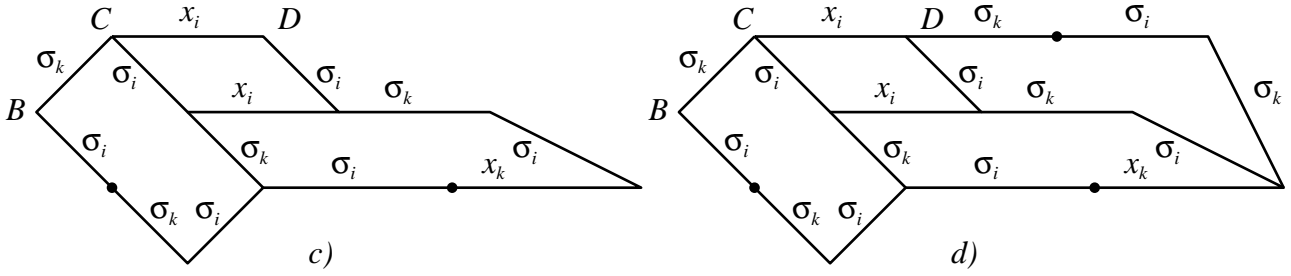


FIGURE 5.

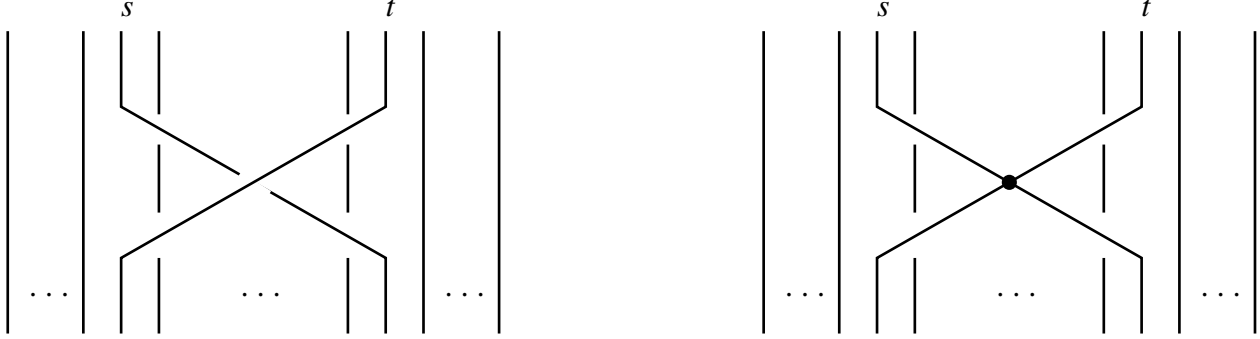


FIGURE 6.

Consider the first case, then using Proposition 1 we obtain the fragment of the Cayley graph, depicted on Figure 3 c). We complete the proof of this case using relation  $\sigma_i \sigma_k x_i = x_k \sigma_i \sigma_k$ . At the last case the edge  $\sigma_i$  starts at  $C$ . Consider the node  $B$  where starts  $\sigma_k$ . If the edge  $\sigma_i$  ends at  $B$  then we use relation  $\sigma_i \sigma_k x_i = x_k \sigma_i \sigma_k$  and this case is finished. So suppose that the edge  $\sigma_i$  also starts at  $B$ . Using Proposition 1 we obtain the fragment of the Cayley graph, depicted on Figure 4 a). Using Proposition 1 two times we obtain the fragment of the Cayley graph, depicted on Figure 4 b). Again using Proposition 1 we come to the fragment of the Cayley graph, depicted on Figure 5 c). To complete the proof of this case we use the relation  $\sigma_i \sigma_k x_i = x_k \sigma_i \sigma_k$  and arrive to Figure 5 d).  $\square$

The definition of the summit set for the singular braid monoid is the same as Garside's.

**Theorem 5.** *In  $SB_n$  two elements are conjugate if and only if their summit sets are identical.*

For any element of  $SB_n$  the summit set is finite and is obtained algorithmically, so this theorem gives a solution of the conjugacy problem.

#### 4. BIRMAN – KO – LEE PRESENTATION FOR THE SINGULAR BRAID MONOID

For the singular braid monoid we prove the existence of the analogue of the presentation of J. S. Birman, K. H. Ko and S. J. Lee (2). For  $1 \leq s < t \leq n$  and  $1 \leq p < q \leq n$  we consider the elements of  $SB_n$  which are defined by

$$(7) \quad \begin{cases} a_{ts} &= (\sigma_{t-1} \sigma_{t-2} \cdots \sigma_{s+1}) \sigma_s (\sigma_{s+1}^{-1} \cdots \sigma_{t-2}^{-1} \sigma_{t-1}^{-1}) \text{ for } 1 \leq s < t \leq n, \\ a_{ts}^{-1} &= (\sigma_{t-1} \sigma_{t-2} \cdots \sigma_{s+1}) \sigma_s^{-1} (\sigma_{s+1}^{-1} \cdots \sigma_{t-2}^{-1} \sigma_{t-1}^{-1}) \text{ for } 1 \leq s < t \leq n, \\ b_{qp} &= (\sigma_{q-1} \sigma_{q-2} \cdots \sigma_{p+1}) x_p (\sigma_{p+1}^{-1} \cdots \sigma_{q-2}^{-1} \sigma_{q-1}^{-1}) \text{ for } 1 \leq p < q \leq n. \end{cases}$$

Geometrically the generators  $a_{s,t}$  and  $b_{s,t}$  are depicted in Figure 6.



**Theorem 6.** *The singular braid monoid  $SB_n$  has a presentation with generators  $a_{ts}$ ,  $a_{ts}^{-1}$  for  $1 \leq s < t \leq n$  and  $b_{qp}$  for  $1 \leq p < q \leq n$  and relations*

$$(8) \quad \begin{cases} a_{ts}a_{rq} = a_{rq}a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} & \text{for } 1 \leq r < s < t \leq n, \\ a_{ts}a_{ts}^{-1} = a_{ts}^{-1}a_{ts} = 1 & \text{for } 1 \leq s < t \leq n, \\ a_{ts}b_{rq} = b_{rq}a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}b_{ts} = b_{ts}a_{ts} & \text{for } 1 \leq s < t \leq n, \\ a_{ts}b_{sr} = b_{tr}a_{ts} & \text{for } 1 \leq r < s < t \leq n, \\ a_{sr}b_{tr} = b_{ts}a_{sr} & \text{for } 1 \leq r < s < t \leq n, \\ a_{tr}b_{ts} = b_{sr}a_{tr} & \text{for } 1 \leq r < s < t \leq n, \\ b_{ts}b_{rq} = b_{rq}b_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0. \end{cases}$$

*Proof.* We follow the proof of J. S. Birman, K. H. Ko and S. J. Lee [5] and we begin with the presentation of  $SB_n$  using generators  $\sigma_i, \sigma_i^{-1}, x_i$ ,  $i = 1, \dots, n-1$ , and relations (3). Add the new generators  $a_{ts}, a_{ts}^{-1}$  for  $1 \leq s < t \leq n$  and  $b_{qp}$  for  $1 \leq p < q \leq n$  and relations (7). Relations (8) are described by isotopies of singular braids, so they must be the consequences of (3), and we may add them too.

In the special case when  $t = s + 1$  relations (7) tell us that  $a_{(s+1)s} = \sigma_s$ ,  $a_{(s+1)s}^{-1} = \sigma_s^{-1}$ ,  $b_{(s+1)s} = x_s$ , so we may omit the generators  $\sigma_1, \dots, \sigma_{n-1}, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}$  to obtain a presentation with generators  $a_{ts}, a_{ts}^{-1}, b_{pq}$ . Defining relations are now (8) and

$$(9) \quad a_{(i+1)i}a_{(j+1)j} = a_{(j+1)j}a_{(i+1)i}, \text{ if } |i-j| > 1,$$

$$(10) \quad a_{(i+1)i}a_{(i+2)(i+1)}a_{(i+1)i} = a_{(i+2)(i+1)}a_{(i+1)i}a_{(i+2)(i+1)},$$

$$(11) \quad a_{ts} = (a_{t(t-1)}a_{(t-1)(t-2)} \cdots a_{(s+2)(s+1)})a_{(s+1)s}(a_{(s+2)(s+1)}^{-1} \cdots a_{(t-1)(t-2)}^{-1}a_{t(t-1)}^{-1})$$

$$(12) \quad a_{ts}^{-1} = (a_{t(t-1)}a_{(t-1)(t-2)} \cdots a_{(s+2)(s+1)})a_{(s+1)s}^{-1}(a_{(s+2)(s+1)}^{-1} \cdots a_{(t-1)(t-2)}^{-1}a_{t(t-1)}^{-1})$$

$$(13) \quad b_{(i+1)i}b_{(j+1)j} = b_{(j+1)j}b_{(i+1)i}, \text{ if } |i-j| > 1,$$

$$(14) \quad b_{(i+1)i}a_{(j+1)j} = a_{(j+1)j}b_{(i+1)i}, \text{ if } |i-j| \neq 1,$$

$$(15) \quad a_{(i+1)i}a_{(i+2)(i+1)}b_{(i+1)i} = b_{(i+2)(i+1)}a_{(i+1)i}a_{(i+2)(i+1)},$$

$$(16) \quad a_{(i+2)(i+1)}a_{(i+1)i}b_{(i+2)(i+1)} = b_{(i+1)i}a_{(i+2)(i+1)}a_{(i+1)i},$$

$$(17) \quad b_{qp} = (a_{q(q-1)}a_{(q-1)(q-2)} \cdots a_{(p+2)(p+1)})b_{(p+1)p}(a_{(p+2)(p+1)}^{-1} \cdots a_{(q-1)(q-2)}^{-1}a_{q(q-1)}^{-1})$$

for  $1 \leq p < q \leq n$ .

Now we prove that relations (9) - (17) are consequences of (8). It was proved by J. S. Birman, K. H. Ko and S. J. Lee that relations (9) - (11) are consequences of the first two relations of (8). The proof for the relation (12) is the same as for (11). Relations (13) are special cases of

the last relations in (8). Relations (14) are special cases of the forth and the fifth relations in (8). To deduce the relation (15) we use at first the sixth relation in (8):

$$a_{(i+1)i}a_{(i+2)(i+1)}b_{(i+1)i} = a_{(i+1)i}b_{(i+2)i}a_{(i+2)(i+1)},$$

and then the seventh relation in (8):

$$a_{(i+1)i}b_{(i+2)i}a_{(i+2)(i+1)} = b_{(i+2)(i+1)}a_{(i+1)i}a_{(i+2)(i+1)}.$$

To deduce the relation (16) we use the second, the eighth, the fifth and again the second relations in (8):

$$\begin{aligned} a_{(i+2)(i+1)}a_{(i+1)i}b_{(i+2)(i+1)} &= a_{(i+1)i}a_{(i+2)i}b_{(i+2)(i+1)} = a_{(i+1)i}b_{(i+1)i}a_{(i+2)i} = \\ &= b_{(i+1)i}a_{(i+1)i}a_{(i+2)i} = b_{(i+1)i}a_{(i+2)(i+1)}a_{(i+1)i}. \end{aligned}$$

To eliminate (17) apply the second relation in (8) to change the center pair in (17)  $a_{(p+2)(p+1)}b_{(p+1)p}$  to  $b_{(p+2)p}a_{(p+1)p}$ . Then apply this process to the pair  $a_{(p+3)(p+2)}b_{(p+2)p}$ . Ultimately, this process will move the original center letter  $b_{(p+1)p}$ , to the leftmost position, where it becomes  $b_{qp}$ . Free cancellation eliminates everything to its right, and we are done.  $\square$

Now we consider the *positive singular braid monoid* with respect to generators  $a_{ts}$  and  $b_{t,s}$  for  $1 \leq s < t \leq n$ . Its relations are (8) except one concerning the invertibility of  $a_{ts}$ . Two positive words  $A$  and  $B$  in the alphabet  $a_{ts}$  and  $b_{t,s}$  will be said to be *positively equivalent* if they are equal as elements of this monoid. In this case, as in the previous section, we shall write  $A \doteq B$ .

The *fundamental word*  $\delta$  of Birman, Ko and Lee is given by the formula

$$\delta \equiv a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{21} \equiv \sigma_{n-1}\sigma_{n-2} \dots \sigma_2\sigma_1.$$

Its divisibility by any generator  $a_{ts}$ , proved in [5], is convenient for us to be expressed in the following form.

**Proposition 7.** *The fundamental word  $\delta$  is positively equivalent to a word that begins or ends with any given generator  $a_{ts}$ . The explicit expression for left divisibility is*

$$\delta \doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}a_{t(t-1)} \dots a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21}.$$

**Proposition 8.** *For the fundamental word  $\delta$  there are the following formulae of commutation*

$$\begin{cases} a_{ts}\delta & \doteq \delta a_{(t+1)(s+1)} \text{ for } 1 \leq r < s < t < n, \\ a_{ns}\delta & \doteq \delta a_{(s+1)1}, \\ b_{ts}\delta & \doteq \delta b_{(t+1)(s+1)} \text{ for } 1 \leq r < s < t < n, \\ b_{ns}\delta & \doteq \delta b_{(s+1)1}. \end{cases}$$

*Proof.* For the generators  $a_{ts}$  it is proved in [5]. Suppose that  $1 \leq r < s < t < n$ , then using relations (8) we have

$$\begin{aligned} b_{ts}\delta &\doteq b_{ts}a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}a_{t(t-1)} \dots a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21} \doteq \\ &= a_{ts}b_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}a_{t(t-1)} \dots a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21} \doteq \\ &= a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots b_{ts}a_{(t+1)s}a_{t(t-1)} \dots a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21} \doteq \\ &= a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}b_{(t+1)t}a_{t(t-1)} \dots a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21} \doteq \\ &= a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}a_{t(t-1)}b_{(t+1)(t-1)} \dots a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21} \doteq \\ &= a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}a_{t(t-1)} \dots b_{(t+1)(s+2)}a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21} \doteq \\ &= a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}a_{t(t-1)} \dots a_{(s+2)(s+1)}b_{(t+1)(s+1)}a_{s(s-1)} \dots a_{21} \doteq \\ &= a_{ts}a_{n(n-1)}a_{(n-1)(n-2)} \dots a_{(t+1)s}a_{t(t-1)} \dots a_{(s+2)(s+1)}a_{s(s-1)} \dots a_{21}b_{(t+1)(s+1)} \doteq \delta b_{(t+1)(s+1)}. \end{aligned}$$



FIGURE 7.

For  $t = n$ , we have

$$\begin{aligned}
 b_{ns}\delta &\doteq b_{ns}a_{n(n-1)}a_{(n-1)(n-2)}\cdots a_{21} \doteq a_{n(n-1)}b_{(n-1)s}a_{(n-1)(n-2)}\cdots a_{21} \doteq \cdots \doteq \\
 &a_{(n-1)n}a_{(n-1)(n-2)}\cdots a_{(s+2)(s+1)}b_{(s+1)s}a_{(s+1)s}\cdots a_{21} \doteq \\
 &a_{(n-1)n}a_{(n-1)(n-2)}\cdots a_{(s+1)s}b_{(s+1)s}a_{s(s-1)}\cdots a_{21} \doteq \\
 &a_{(n-1)n}a_{(n-1)(n-2)}\cdots a_{s(s-1)}b_{(s+1)(s-1)}a_{(s-1)(s-2)}\cdots a_{21} \doteq \\
 &a_{(n-1)n}a_{(n-1)(n-2)}\cdots a_{s(s-1)}a_{(s-1)(s-2)}\cdots a_{21}b_{(s+1)1}.
 \end{aligned}$$

□

Geometrically this commutation is shown on Figure 7.

The analogues of the other results proved by Birman, Ko and Lee (which are also analogues of the statements of Sections 2 and 3 of the present paper) remain valid for the singular braid monoid. Unfortunately the proof of the analogue of Proposition 1 (which consists of many different cases already in [5]) is very long and consists of even bigger amount of cases.

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